Séminaire Lotharingien de Combinatoire **78B** (2017)
Article #38, 11 pp.

Proceedings of the 29th Conference on Formal Power Series and Algebraic Combinatorics (London)

Torus link homology and the nabla operator

Andrew Timothy Wilson*

Department of Mathematics, University of Pennsylvania

Abstract. In recent work, Elias and Hogancamp develop a recurrence for the Poincaré series of the triply graded Hochschild homology of certain links, one of which is the (n, n) torus link. In this case, Elias and Hogancamp give a combinatorial formula for this homology that is reminiscent of the combinatorics of the modified Macdonald polynomial eigenoperator ∇. We give a combinatorial formula for the homologies of all links considered by Elias and Hogancamp. Our first formula is not easily computable, so we show how to transform it into a computable version. Finally, we conjecture a direct relationship between the (n, n) torus link case of our formula and the symmetric function ∇p_{1^n}.

Keywords: torus links, symmetric functions, Macdonald polynomials, nabla operator

1 Introduction

We begin by establishing some notation from knot theory, following [6]. The remaining sections of the paper will take a more combinatorial perspective.

The *braid group on n strands*, denoted Br_n, can be defined by the presentation

$$\mathrm{Br}_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,|\, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},\, \sigma_i \sigma_j = \sigma_j \sigma_i \right\rangle \tag{1.1}$$

for all $1 \leq i \leq n-2$ and $|i-j| \geq 2$. This group can be pictured as all ways to "braid" together n strands, where σ_i corresponds to crossing string $i+1$ over string i and the group operation is concatenation. One particularly notable braid is the *full twist braid* on n strands, denoted FT_n, which can be written

$$\mathrm{FT}_n = \left((\sigma_1)(\sigma_2\sigma_1) \ldots (\sigma_{n-1}\sigma_{n-2} \ldots \sigma_1) \right)^2 . \tag{1.2}$$

where multiplication is left to right. We will also need an operation ω on braids which corresponds to rotation around the horizontal axis. We define ω on Br_n by $\omega(\sigma_i) = \sigma_i$ and $\omega(\alpha\beta) = \omega(\beta)\omega(\alpha)$. Then ω is an anti-involution on Br_n. All of our braids will have the property that the string that begins in column i also ends in column i for all i; these are sometimes called *perfect braids*.

*andwils@math.upenn.edu. The author was supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship.

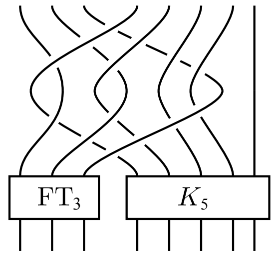


Figure 1: We have drawn the complex $C_{10101101}$, where FT_3 is the full twist braid and K_5 is a certain complex defined recursively in [6]. This figure is used courtesy of [6].

Given a braid with n strands, one can form a *link* (i.e. nonintersecting collection of knots) by identifying the top of the strand that begins in position i with the bottom of the strand that ends in position i for $1 \leq i \leq n$. The result is called a *closed braid*. Alexander proved that every link can be represented by a closed braid (although this representation is not unique) [1]. The closure of a perfect braid is a link that consists of n separate unknots linked together.

In [6], Elias and Hogancamp assign a complex C_v to every binary word v. We describe this assignment here – see Figure 1 for an example. Say $v \in \{0,1\}^n$ with $|v| = m$. We begin with two braids, the full twist braid FT_{n-m} and a certain recursively defined complex K_m [6], which sits to the right of FT_{n-m}. For $i = 1$ to n, we feed string i into the leftmost available position in K_m if $v_i = 1$; otherwise, we feed string i into the leftmost available position in FT_{n-m}. All crossings that occur are forced to be "positive," i.e. the right strand crosses over the left strand. This induces a braid $\beta_v \in \text{Br}_n$ that occurs before the adjacent FT_{n-m} and K_m. The final complex C_v is obtained by performing $\omega(\beta_v)$, followed by β_v, followed by the adjacent FT_{n-m} and K_m. We note that C_{0^n} is the full twist braid FT_n and that the closure of this braid is the (n,n) torus link. The combinatorics of other links, in particular the (m,n) torus link for m and n coprime, has been studied by a variety of authors in recent years [8, 7]. Haglund gives an overview of this work from a combinatorial perspective in [10].

Elias and Hogancamp map each complex C_v to a graded Soergel bimodule and then consider the *Hochschild homology* of this bimodule; this is sometimes called Khovanov-Rozansky homology [12, 13]. This homology has three gradings: the bimodule degree (using the variable Q), the homological degree (T), and the Hochschild degree (A). After the grading shifts $q = Q^2$, $t = T^2Q^{-2}$, and $a = AQ^{-2}$, Elias and Hogancamp give a recurrence for the Poincaré series of this triply graded homology, which they denote $f_v(q,a,t)$. They also give a combinatorial formula for the special case $f_{0^n}(q,a,t)$. We will give two combinatorial formulas for $f_v(q,a,t)$ for every $v \in \{0,1\}^n$.

In Section 2, we define a symmetric function $L_v(x;q,t)$ which we call the *link symmetric function*. Its definition is reminiscent of the combinatorics of the Macdonald eigenoperator ∇, introduced in [3]. We prove that $f_v(q,a,t)$ is equal to a certain inner product with $L_v(x;q,t)$.

The main weakness of our first formula is that it is a sum over infinitely many objects, so it is not clear how to compute using this formula. We address this issue in Section 3, obtaining a finite formula for $L_v(x;q,t)$ using a collection of combinatorial objects we call *barred Fubini words*.

We close by presenting some conjectures in Section 4. In particular, we conjecture that

$$L_{0^n}(x;q,t) = (1-q)^{-n}\nabla p_{1^n}. \tag{1.3}$$

where the terminology is defined in Section 4. A proof of this conjecture would provide the first combinatorial interpretation for ∇p_{1^n}. There has been much recent work establishing combinatorial interpretations for ∇e_n [5] and ∇p_n [15]. We believe that $L_v(x;q,t)$ is also related to Macdonald polynomials for general v, although we do not have an explicit conjecture in this direction.

2 An infinite formula

Let $\mathbb{N} = \{0,1,2,\ldots\}$ and $\mathbb{Z}_+ = \{1,2,3,\ldots\}$. We begin by defining two statistics.

Definition 2.1. *Given words $\gamma \in \mathbb{N}^n$ and $\pi \in \mathbb{Z}_+^n$, we define*

$$\operatorname{area}(\gamma) = |\gamma| - \#\{1 \le i \le n : \gamma_i > 0\} \tag{2.1}$$

$$\begin{aligned}\operatorname{dinv}(\gamma,\pi) = &\#\{1 \le i < j \le n : \gamma_i = \gamma_j, \pi_i > \pi_j\} \\ &+ \#\{1 \le i < j \le n : \gamma_i + 1 = \gamma_j, \pi_i < \pi_j\}\end{aligned} \tag{2.2}$$

$$x^\pi = \prod_{i=1}^n x_{\pi_i}. \tag{2.3}$$

In Figure 2, we draw a diagram for $\gamma = 20141022$ and $\pi = 41322231$. Area counts the empty boxes in such a diagram, dinv counts certain pairs of labels, and x^π records all labels that appear in the diagram.

Definition 2.2. *Given $n \in \mathbb{Z}_+$ and $v \in \{0,1\}^n$, define*

$$L_v = L_v(x;q,t) = \sum_{\substack{\gamma \in \mathbb{N}^n,\, \pi \in \mathbb{Z}_+^n \\ \gamma_i = 0 \Leftrightarrow v_i = 1}} q^{\operatorname{area}(\gamma)} t^{\operatorname{dinv}(\gamma,\pi)} x^\pi. \tag{2.4}$$

Perhaps the first thing to note about L_v is that it is symmetric in the x variables; one way to see this is to express L_v as a sum of LLT polynomials [14]. More precisely, each $\gamma \in \mathbb{N}^n$ can be associated with an n-tuple $\boldsymbol{\lambda}(\gamma)$ of single cell partitions in the plane,

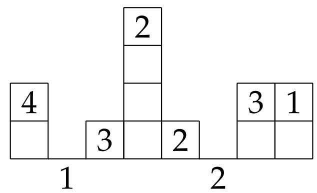


Figure 2: We have depicted the example $\gamma = 20141022$ and $\pi = 41322231$ by drawing bottom-justified columns with heights $\gamma_1, \gamma_2, \ldots, \gamma_8$ and the labels π_i are placed as high as possible in each column. In this example, we compute $\operatorname{area}(\gamma) = 6$, $\operatorname{dinv}(\gamma, \pi) = 7$, where the contributing pairs are in columns $(1,7)$, $(1,8)$, $(2,3)$, $(2,5)$, $(3,5)$, $(5,7)$, $(7,8)$, and $x^\pi = x_1^2 x_2^3 x_3^2 x_4$.

where the ith cell is placed on diagonal γ_i and the order is not changed. Using the notation of [11], the unicellular LLT polynomial $G_{\lambda(\gamma)}(x;t)$ can be used to write

$$L_v = \sum_{\substack{\gamma \in \mathbb{N}^n \\ \gamma_i = 0 \Leftrightarrow v_i = 1}} q^{\operatorname{area}(\gamma)} G_{\lambda(\gamma)}(x;t). \tag{2.5}$$

Since LLT polynomials are symmetric, every L_v is also symmetric.

We also remark that L_{1^n} is equal to the modified Macdonald polynomial $\widetilde{H}_{1^n}(x;q,t)$, which is also equal to the graded Frobenius series of the coinvariants of $\mathfrak{S}_n$ with grading in t.

Next, we note that the Poincaré series $f_v(q,a,t)$ can be recovered as a certain inner product of L_v. We follow the standard notation for symmetric functions and their usual inner product, as described in Chapter 7 of [19]. The following lemma leads to Theorem 2.1. We omit the proofs from this extended abstract.

Lemma 2.1.

$$L_{0^n} = \frac{1}{1-q} L_{10^{n-1}}. \tag{2.6}$$

Theorem 2.1. *For any $v \in \{0,1\}^n$,*

$$f_v(q,a,t) = \sum_{d=0}^{n} \langle L_v, e_{n-d} h_d \rangle \, a^d. \tag{2.7}$$

For the sake of comparison with [6], we give a simplified formula that directly computes $f_v(q,a,t)$ from Theorem 2.1. Given $\gamma \in \mathbb{N}^n$ and $1 \le i \le n$, let

$$\operatorname{dinv}_i(\gamma) = \#\{j < i : \gamma_j = \gamma_i\} + \#\{j > i : \gamma_j = \gamma_i + 1\}. \tag{2.8}$$

Corollary 2.1.

$$f_v(q,a,t) = \sum_{\substack{\gamma \in \mathbb{N}^n \\ \gamma_i = 0 \Leftrightarrow v_i = 1}} q^{\mathrm{area}(\gamma)} \prod_{i=1}^{n} \left(a + t^{\mathrm{dinv}_i(\gamma)} \right) \tag{2.9}$$

where, as before, $\mathrm{area}(\gamma) = |\gamma| - \#\{1 \le i \le n : \gamma_i > 0\}$.

If $v = 0^n$ and $a = 0$, this is exactly Theorem 1.9 in [6].

Let us use Corollary 2.1 to compute $f_{00}(q,a,t)$. We consider three cases: either $\gamma_2 \in \{\gamma_1, \gamma_1 + 1\}$, $\gamma_2 > \gamma_1 + 1$, or $\gamma_2 < \gamma_1$. In the first case, we get a contribution of $(1 + a)(a + t)$ from the dinv product in Corollary 3.1; in the other cases, we get $(1 + a)^2$.

In the first case, our minimum area is either 0 or 1, depending on if $\gamma_2 = \gamma_1$ or $\gamma_1 + 1$. Either way, we can choose any nonnegative integer k and add k to both γ_1 and γ_2. This yields an area contribution of $(1 + q)(1 - q^2)$. In the second case, the minimum area is 2 and not only can we add area to both γ_1 and γ_2 equally but we can add area to γ_2 by itself. The resulting contribution is $q^2/((1 - q^2)(1 - q))$. Similarly, in the third case we get a contribution of $q/((1 - q^2)(1 - q))$.

Summing up all these weights, we get

$$f_{00}(q,a,t) = \frac{(1+q)(1+a)(a+t)}{1-q^2} + \frac{q^2(1+a)^2}{(1-q^2)(1-q)} + \frac{q(1+a)^2}{(1-q^2)(1-q)} \tag{2.10}$$

$$= \frac{q + t - qt + a(1 + q + t - qt) + a^2}{(1-q)^2}. \tag{2.11}$$

This process can become difficult for v with length as short as 3. The difficulty comes mostly from considering how to add (possibly infinite amounts of) area without affecting the dinv vectors. We address this issue in the next section by deriving a formula that allows us to restrict to a finite set of γ vectors.

3 A finite formula

Although the combinatorial definition of L_v is straightforward, it is not computationally effective[2] since it is a sum over infinitely many words $\gamma \in \mathbb{N}^n$. We rectify this issue in Theorem 3.1 below. The idea is to compress the vectors γ while altering the statistics so that the link polynomial L_v is not changed.

Definition 3.1. *A word* $\gamma \in \mathbb{N}^n$ *is a* Fubini word *if every integer* $0 \le k \le \max(\gamma)$ *appears in* γ.

[2] There are also infinitely many $\pi \in \mathbb{Z}_+^n$, but this problem can be addressed with standardization [9].

For example, 41255103 is a Fubini word but 20141022 is not a Fubini word, since it contains a 4 but not a 3. We call these Fubini words because they are counted by the Fubini numbers ([18], A000670), which also count ordered partitions of the set $\{1, 2, \ldots, n\}$. We will actually be interested in certain decorated Fubini words.

Definition 3.2. *Given $v \in \{0,1\}^n$, we say that a Fubini word γ is* associated *with v if either*

- $v = 0^n$ *and the only zero in γ occurs at γ_1, or*
- $v \neq 0^n$ *and $\gamma_i = 0$ if and only if $v_i = 1$.*

Definition 3.3. *A* barred Fubini word *associated with v is a Fubini word γ associated with v where we may place bars over certain entries. Specifically, the entry γ_j may be barred if*

1. $\gamma_j > 0$,
2. γ_j *is unique in γ, and*
3. *for each $i < j$ we have $\gamma_i < \gamma_j$, i.e. γ_j is a left-to-right maximum in γ.*

We denote the collection of barred Fubini words associated with v by $\overline{\mathcal{F}}_v$.

For example,

$$\overline{\mathcal{F}}_0 = \{0\} \tag{3.1}$$
$$\overline{\mathcal{F}}_{00} = \{01, 0\overline{1}\} \tag{3.2}$$
$$\overline{\mathcal{F}}_{000} = \{011, 012, 0\overline{1}2, 01\overline{2}, 0\overline{1}\overline{2}, 021, 0\overline{2}1\}. \tag{3.3}$$

The sequence $|\overline{\mathcal{F}}_{0^n}|$ for $n \in \mathbb{N}$ begins $1, 1, 2, 7, 35, 226, \ldots$ and seems to appear in the OEIS as A014307 [18]. One way to define sequence A014307 is that it has exponential generating function

$$\sqrt{\frac{e^z}{2 - e^z}}. \tag{3.4}$$

This sequence is given several combinatorial interpretations in [17]. It would be interesting to obtain a bijection between $\overline{\mathcal{F}}_{0^n}$ and one of the collections of objects in [17]. See Figure 3 for more examples of barred Fubini words.

Given a barred Fubini word γ and a word $\pi \in \mathbb{Z}_+^n$, we use the same area statistic and modify the dinv statistic slightly:

$$\operatorname{area}(\gamma) = |\gamma| - \#\{1 \leq i \leq n : \gamma_i > 0\} \tag{3.5}$$
$$\operatorname{dinv}(\gamma, \pi) = \#\{1 \leq i < j \leq n : \gamma_i = \gamma_j, \pi_i > \pi_j\} \tag{3.6}$$
$$+ \#\{1 \leq i < j \leq n : \gamma_i + 1 = \gamma_j, \pi_i < \pi_j, \gamma_j \text{ is not barred}\}.$$

We also let $\operatorname{bar}(\gamma)$ be the number of barred entries in γ. We have the following result.

v	$\overline{\mathcal{F}}_v$
111	000
011	$100, \bar{1}00$
101	$010, 0\bar{1}0$
110	$001, 00\bar{1}$
001	$110, 120, 1\bar{2}0, \bar{1}20, \bar{1}\bar{2}0, 210, \bar{2}10$
010	$101, 102, 10\bar{2}, \bar{1}02, \bar{1}0\bar{2}, 201, \bar{2}01$
100	$011, 012, 0\bar{1}2, 01\bar{2}, 0\bar{1}\bar{2}, 021, 0\bar{2}1$
000	$011, 012, 0\bar{1}2, 01\bar{2}, 0\bar{1}\bar{2}, 021, 0\bar{2}1$

Figure 3: We have listed the barred Fubini words $\overline{\mathcal{F}}_v$ for each $v \in \{0,1\}^3$.

Theorem 3.1. *For $v \in \{0,1\}^n$,*

$$L_v = \sum_{\substack{\gamma \in \overline{\mathcal{F}}_v \\ \pi \in \mathbb{Z}_+^n}} q^{\operatorname{area}(\gamma)+\operatorname{bar}(\gamma)} t^{\operatorname{dinv}(\gamma,\pi)} (1-q)^{-\operatorname{bar}(\gamma)-\chi(v=0^n)} x^\pi \tag{3.7}$$

where χ of a statement is 1 if the statement is true and 0 if it is false.

As in Section 2, we give a formula for computing $f_v(q,a,t)$ directly. Given a barred Fubini word γ, we define

$$\operatorname{dinv}_i(\gamma) = \#\{j < i : \gamma_j = \gamma_i\} + \#\{j > i : \gamma_j = \gamma_i + 1, \gamma_j \text{ is not barred}\}. \tag{3.8}$$

Corollary 3.1.

$$f_v(q,a,t) = \sum_{\gamma \in \overline{\mathcal{F}}_v} q^{\operatorname{area}(\gamma)+\operatorname{bar}(\gamma)} (1-q)^{-\operatorname{bar}(\gamma)-\chi(v=0^n)} \prod_{i=1}^{n} \left(a + t^{\operatorname{dinv}_i(\gamma)}\right) \tag{3.9}$$

Let us use Corollary 3.1 to compute $f_{00}(q,a,t)$ again. We note that $\overline{\mathcal{F}}_{00} = \{01, 0\bar{1}\}$. Computing the weights for these vectors and then adding, we get

$$f_{00}(q,a,t) = \frac{(1+a)(a+t)}{1-q} + \frac{q(1+a)^2}{(1-q)^2} \tag{3.10}$$

$$= \frac{q+t-qt+a(1+q+t-qt)+a^2}{(1-q)^2} \tag{3.11}$$

which is the same answer we computed at the end of Section 2.

t^2			
t	qt	q^2t	
1	q	q^2	q^3

Figure 4: This is the Ferrers diagram of the partition $\mu = (4, 3, 1)$. In each cell we have written the monomial $q^i t^j$ that corresponds to the cell, yielding $B_\mu = \{1, q, q^2, q^3, t, qt, q^2t, t^2\}$.

4 Conjectures

So far, we have used the inner product $\langle L_v, e_{n-d}h_d \rangle$ to compute $f_v(q, a, t)$; one might wonder if there is any value in studying the full symmetric function L_v. In this section, we conjecture that the link symmetric function L_v is closely related to the combinatorics of Macdonald polynomials, hinting at a stronger connection between Macdonald polynomials and link homology. Following [6], we must first define a "normalized" version of the link symmetric function L_v.

Definition 4.1.

$$\widetilde{L}_v = \widetilde{L}_v(x; q, t) = (1-q)^{n-|v|} L_v(x; q, t). \tag{4.1}$$

We could also define $\widetilde{L}_v$ in terms of diagrams; each box that contains a label from π contributes an additional factor of $1 - q$. Theorem 3.1 implies that $\widetilde{L}_v$ has coefficients in $\mathbb{Z}[q, t]$, whereas the coefficients of L_v are elements of $\mathbb{Z}[[q, t]]$. We conjecture that the normalized link symmetric function $\widetilde{L}_v$ is closely connected to the Macdonald eigenoperators ∇ and Δ.

The modified Macdonald polynomials $\widetilde{H}_\mu$ form a basis for the ring of symmetric functions with coefficients in $\mathbb{Q}(q, t)$. They can be defined via triangularity relations or combinatorially [11, 9]. Given a partition μ, let B_μ be the alphabet of monomials $q^i t^j$ where (i, j) ranges over the coordinates of the cells in the Ferrers diagram of μ. We compute an example in Figure 4.

Given a symmetric function F and a set of monomials $A = \{a_1, a_2, \ldots, a_n\}$, we let $F[A]$ be the result of setting $x_i = a_i$ for $1 \leq i \leq n$ and $x_i = 0$ for $i > n$. Then we define two operators on symmetric functions by setting, for $\mu \vdash n$,

$$\Delta_F \widetilde{H}_\mu = F\left[B_\mu\right] \widetilde{H}_\mu \tag{4.2}$$

$$\nabla \widetilde{H}_\mu = \Delta_{e_n} \widetilde{H}_\mu \tag{4.3}$$

and expanding linearly. Note that, for $\mu \vdash n$, $e_n[B_\mu]$ is simply the product of the n monomials in B_μ, which is sometime written T_μ in the literature.

Conjecture 4.1.

$$\nabla p_{1^n} = \widetilde{L}_{0^n} \tag{4.4}$$

$$\Delta_{e_{n-1}} p_{1^n} = \sum_{\substack{v \in \{0,1\}^n \\ |v|=1}} \widetilde{L}_v \tag{4.5}$$

In fact, both conjectures follow from the conjecture that

$$\widetilde{L}_{v0} = \nabla p_1 \nabla^{-1} \widetilde{L}_v. \tag{4.6}$$

We should mention that Eugene Gorsky first noticed that the identity

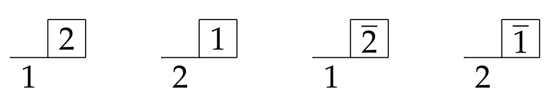
$$\sum_{a=0}^{d} \langle \nabla p_{1^n}, e_{n-d} h_d \rangle \, a^d = (1-q)^n f_{0^n}(q,a,t) \tag{4.7}$$

seemed to hold and communicated this observation to the author via Jim Haglund. Gorsky's conjectured identity is a special case of Conjecture 4.1. It is also interesting to note that the operator in (4.6) appears in the setting of the Rational Shuffle Conjecture as $-\mathbf{Q}_{1,1}$ [4].

As an example of our conjecture, we can use Sage to compute

$$\langle \nabla p_{1,1}, p_{1,1} \rangle = 1 + q + t - qt. \tag{4.8}$$

This expression should equal $\left\langle \widetilde{L}_{00}, p_{1,1} \right\rangle$ by Conjecture 4.1. To compute this inner product using Theorem 3.1, we consider the barred Fubini words 01 and $0\overline{1}$, each of which can receive labels $\pi = 12$ or 21. The corresponding diagrams are



where we have moved the bars from γ_i to the corresponding π_i. The weights of these diagrams coming from Theorem 3.1 are

$$\frac{t}{1-q} \qquad \frac{1}{1-q} \qquad \frac{q}{(1-q)^2} \qquad \frac{q}{(1-q)^2} \tag{4.9}$$

respectively. After multiplying by the normalizing factor $(1-q)^2$ to go from L_{00} to $\widetilde{L}_{00}$, we sum the resulting weights to get

$$(1-q)t + 1 - q + q + q = 1 + q + t - qt \tag{4.10}$$

as desired.

After reading an earlier version of this paper, François Bergeron contacted the author with the following additional conjectures.

Conjecture 4.2 (Bergeron, 2016)**.**

$$L_{v0} = L_{1v} + qL_{0v} \tag{4.11}$$

$$L_{0^n} = \sum_{v \in \{0,1\}^k} q^{n-|v|} L_{v0^{n-k}} \tag{4.12}$$

$$t\left(L_{u011v} - L_{u101v}\right) = L_{u101v} - L_{u110v} \tag{4.13}$$

$$\widetilde{L}_{0^a1^b0^c} = \nabla p_{1^c} \nabla^{-1} \widetilde{H}_{1^b} \nabla p_{1^a} \tag{4.14}$$

$$L_{1^a01^b} = \frac{t^a - 1}{t^{a+b} - 1} \left[\nabla p_1 \nabla^{-1}, \widetilde{H}_{1^{a+b}}\right] + \widetilde{H}_{1^{a+b}} p_1 \tag{4.15}$$

where the bracket represents the Lie bracket and operators are applied to 1 if nothing is explicitly specified. Bergeron also observed that $L_v(x; q, 1+t)$ is e-positive. (For more context on this last statement, see Section 4 of [2].)

It is clear that (4.11) implies (4.12). We do not know of any other relations between these conjectures. We close with more open questions.

1. Is there a Macdonald eigenoperator expression for $\widetilde{L}_v$ for other v? Perhaps we can use ideas from the Rational Shuffle Conjecture [4], recently proved by Mellit [16]. In particular, Mellit's proof relies heavily on facts about toric braids, which seems relevant to the appearance of torus links in our setting.

2. Can we generalize our conjecture for ∇p_{1^n} to "interpolate" between our conjecture and the Shuffle Theorem [5], or maybe the Square Paths Theorem [15]?

5 Acknowledgements

The author would like to Ben Elias and Matt Hogancamp for their exciting paper and for use of Figure 1; Lyla Fadali for reading an earlier draft; Jim Haglund for editing and feedback; Eugene Gorsky for his comments and for the idea that Elias and Hogancamp's work could be related to Macdonald polynomials; and François Bergeron for Conjecture 4.2 along with other helpful suggestions.

References

[1] J. Alexander. "A lemma on systems of knotted curves". *Proc. Nat. Acad. Sci.* **9** (1923), pp. 93–95. DOI.

[2] F. Bergeron. "Open Questions for operators related to Rectangular Catalan Combinatorics". 2016. arXiv:1603.04476.

[3] F. Bergeron, A. M. Garsia, M. Haiman, and G. Tesler. "Identities and positivity conjectures for some remarkable operators in the theory of symmetric functions". *Meths. Appls. Analysis* **6** (1999), pp. 363–420. DOI.

[4] F. Bergeron, A. Garsia, E. S. Leven, and G. Xin. "Compositional (km, kn)-Shuffle Conjectures". *Int. Math. Res. Notices* **2016** (2016), pp. 4229–4270. DOI.

[5] E. Carlsson and A. Mellit. "A proof of the shuffle conjecture". 2015. arXiv:1508.06239.

[6] B. Elias and M. Hogancamp. "On the computation of the torus link homology". 2016. arXiv:1603.00407.

[7] E. Gorsky and A. Neguţ. "Refined knot invariants and Hilbert schemes". *J. Math. Pures Appl.* **104** (2015), pp. 403–435. DOI.

[8] E. Gorsky, A. Oblomkov, J. Rasmussen, and V. Shende. "Torus knots and the rational DAHA". *Duke Math. J.* **163** (2014), pp. 2709–2794. DOI.

[9] J. Haglund. *The q,t-Catalan Numbers and the Space of Diagonal Harmonics*. University Lecture Series, Vol. 41. Amer. Math. Soc., 2008.

[10] J. Haglund. "The combinatorics of knot invariants arising from the study of Macdonald polynomials". *Recent Trends in Combinatorics*. The IMA Volumes in Mathematics and its Applications. Springer, 2016, pp. 579–600.

[11] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for Macdonald polynomials". *J. Amer. Math. Soc.* **18** (2005), pp. 735–761. DOI.

[12] M. Khovanov. "Triply-graded link homology and Hochschild homology of Soergel bimodules". *Internat. J. Math.* **18** (2007), pp. 869–885. DOI.

[13] M. Khovanov and L. Rozansky. "Matrix factorizations and link homology". *Fund. Math.* **199** (2008), pp. 1–91. DOI.

[14] A. Lascoux, B. Leclerc, and J.-Y. Thibon. "Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties". *J. Math. Phys.* **38** (1997), pp. 1041–1068. DOI.

[15] E. S. Leven. "A proof of the Square Paths Conjecture". 2016. arXiv:1601.06249.

[16] A. Mellit. "Toric braids and (m, n)-parking functions". 2016. arXiv:1604.07456.

[17] Q. Ren. "Ordered partitions and drawings of rooted plane trees". *Discrete Math.* **338** (2015), pp. 1–9. DOI.

[18] N. J. A. Sloane. "The On-Line Encyclopedia of Integer Sequences". URL.

[19] R. P. Stanley. *Enumerative Combinatorics*. Vol. 2. Cambridge University Press, 1999.